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# Theoretical Analysis of Performance Parameters in Oscillating Plasma Thrusters

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**Conventional expressions and definitions describing performance of plasma thrusters, including the thrust, specific impulse, and the thruster efficiency, assume a steady-state plasma flow with a constant flow velocity. However, it is very common for these thrusters that the plasma exhibits unstable behavior resulting in time variations of the thrust and the exhaust velocity. For example, in Hall thrusters, the ionization instability leads to strong oscillations of the discharge current (so-called breathing oscillations), plasma density, ion energy, and as a result the ion flow. This paper revisits the formulation of the thrust and the thrust efficiency to account for time variations of the ion parameters, including the phase shift between the ion energy and the ion flow. For sinusoidal oscillations it was found that thrust can potentially change more than 20%. It is shown that, by modulating ion energy at specific amplitudes, thrust can be maximized in such regimes. Finally, an expression for the thruster efficiency of the modulating thruster is derived to show a mechanism for inefficiencies in such thrusters.**



## Nomenclature

| | | |
|---|---|---|
| $a$ | = | coefficient of series expansion |
| $e$ | = | charge, C |
| $I$ | = | current, A |
| $Isp$ | = | specific impulse, s |
| $\bar{i}$ | = | normalized current |
| $K$ | = | kinetic energy, J |
| $M$ | = | mass of ion, kg |
| $\dot{m}$ | = | mass flow, kg/s |
| $P_{\text{in}}$ | = | input electrical power, W |
| $P_{\text{Kinetic}}$ | = | kinetic power, W |
| $P_{\text{Thrust}}$ | = | thrust power, W |
| $T$ | = | thrust, N |
| $T_D$ | = | decreasing thrust portion, N |
| $T_I$ | = | increasing thrust portion, N |
| $T_{\text{steady}}$ | = | steady-state thrust, N |
| $V$ | = | potential difference, V |
| $v_{\text{ex}}$ | = | exhaust velocity, m/s |
| $v_{\text{jet}}$ | = | jet velocity, m/s |
| $\alpha$ | = | duty cycle of ion energy square wave |
| $\beta$ | = | duty cycle of ion current square wave |
| $\eta$ | = | total efficiency |
| $\eta_{\text{curr}}$ | = | current efficiency |
| $\eta_d$ | = | dispersion efficiency |
| $\eta_{\text{osc}}$ | = | oscillation factor |
| $\eta_{\text{prop}}$ | = | propellant utilization |
| $\eta_{\text{volt}}$ | = | voltage efficiency |
| $\tau$ | = | period of oscillation, s |
| $\bar{v}$ | = | normalized potentials |
| $\phi$ | = | phase difference, rad |

*Subscripts*

| | | |
|---|---|---|
| $a$ | = | amplitude |
| $d$ | = | discharge |
| $H$ | = | high |
| $i$ | = | ion |
| $L$ | = | low |
| $m$ | = | mean |
| $n$ | = | neutral |
| tot | = | total |

## I. Introduction

FOR many plasma thrusters, plasma instabilities often result in unstable thruster operation, which may affect thruster performance and lifetime. For example, operation of Hall thrusters often exhibits discharge current oscillations of $\sim 10^4$ Hz [1–4]. These discharge oscillations result in oscillations of the plasma flow from the thruster. Apart from these naturally occurring oscillations, recent studies have explored externally driven oscillations of the discharge to control the thruster operation. This concept has been studied theoretically in relation to ion sources, where the effect of such oscillations has been suggested to increase current density limits [5]. For Hall thrusters this idea was explored experimentally by measuring the ion current and thrust while modulating the anode voltage [6–9]. In particular, work by Romadanov et al. modulated the direct current (DC) discharge voltage with a sinusoidal signal. It was found that thrust under such oscillating regimes may degrade when the ion flow and ion energy oscillations shift out of phase. In addition, these experiments revealed that this phase shift becomes appreciable, particularly at large oscillation amplitudes [10]. Such phase shifts may exist in any plasma thruster with largely monoenergetic ions, such as ion thrusters, Hall thrusters, and Magnetoplasmadynamic thrusters. A key question then is what effect this phase shift may have on thruster performance, including the thrust, specific impulse (momentum transfer per unit mass of propellant), and thruster efficiency. To the best of our knowledge this question has never been addressed in plasma thruster literature. Therefore, the main goal of this paper is to derive and compare time-resolved and time-averaged thrust and thruster efficiency expressions and relative values for oscillating plasma thrusters.

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This paper is organized as follows. Section II discusses performance for a steady-state plasma thruster operation and shows the importance of accounting for time-dependent oscillations in the formulation of the thruster performance. Section III derives thrust for a sinusoidal modulated thruster. Section IV introduces the expressions for the input power under modulation and the performance of modulated thrusters. Finally dispersion efficiency in the oscillating plasma thruster is discussed in Sec. V.

## II. From Steady State to Time Dependent

### A. Remarks on Steady-State Thruster Performance

For a steady-state operation of the thruster, the thrust is defined as

$$T = \dot{m}_{\text{tot}} v_{\text{jet}} \tag{1}$$

where $\dot{m}_{\text{tot}}$ is the total mass flow and $v_{\text{jet}}$ is the jet velocity. The efficiency is defined here as the ratio between the thrust power $P_{\text{thrust}}$ and the input electrical power to the thruster anode $P_{\text{in}}$, and does not include power through other components such as coils or cathodes:

$$\eta = \frac{P_{\text{thrust}}}{P_{\text{in}}} = \frac{T^2}{2\dot{m}_{\text{tot}} P_{\text{in}}} \tag{2}$$

$$P_{\text{in}} = I_d V_d \tag{3}$$

where $I_d$ and $V_d$ are discharge current and voltage, respectively. When the plasma flow and the plasma exhaust velocity oscillate about a mean value, the derivation of the thrust becomes much more involved. For the sake of our analysis, we consider the time-dependent thrust produced by ion acceleration in an applied electric field. This mechanism of ion acceleration is relevant to Hall thrusters and ion thrusters. Detailed descriptions of how the electric field is generated can be found in other references [11]. The time dependence of the thrust is due to either natural plasma oscillations or external modulations of the applied power or voltage. Note that this analysis may be applicable to thrusters that generate thrust through other means; however, the ion energy distributions of such devices tend to be wider and here we are assuming monoenergetic ions.

### B. Transient Time Scales

Consider a plasma thruster operating in a pulsed mode where the frequency of pulse repetition is much slower than the time scale of the transient plasma processes, such as a breathing instability oscillating the ion current. For such regimes, the derivation of the time-dependent performance is relatively simple. Because of the negligible timescales of the transient processes, the accelerating voltage (energy/charge) and mass flow are in phase (Fig. 1) and so the thrust is dependent only on the duty cycle ($\alpha$) of the pulsing. Pulsed thrust can be expressed as

$$T = \frac{1}{\tau}\int_0^{\tau} \dot{m}(t) v_{\text{ex}}(t)\,\mathrm{d}t \tag{4}$$

$$T_{\text{pulsed}} = \sqrt{\frac{2M}{e}}\left(I_L\sqrt{V_L}(1-\alpha) + I_H\sqrt{V_H}\alpha\right) \tag{5}$$

where $\tau$ is the period of the pulse; $I_L$ and $I_H$ are the low and high levels of the ion current, respectively; $V_L$ and $V_H$ are the low and high levels of the ion energy, respectively; and $\alpha$ is the duty cycle of the square-wave pulse.

Consider now that the repetition frequency increases to approach the transient time scales of the thruster (e.g., the frequency of natural oscillations or the pulse duration). Under such conditions, nonlinear thruster plasma response (e.g., resonance-kind behavior [6] or a hysteresis) can cause the ion current to lag or lead the ion energy, and in some cases, alter the shape of the waveforms. Limiting the current analysis to square waves, the performance should depend on both the duty cycles of each and their phase shift $\phi_i$. This causes the relatively simple square pulse case to be considerably more difficult as there can be multiple solutions depending on the relative duty cycles and phases of the ion energy and ion current flow. Appendix A contains the exact solutions of the thrust for a high-frequency square-wave oscillation for each of these six cases.

While it is not difficult to derive, the sheer variety of solutions for thrust for the various square-wave forms is inconvenient for usage. Therefore, for the rest of the paper, a sine-wave oscillation is considered for two reasons. The first is that there is a single solution to the time-dependent thrust and power, and the second is that this solution is directly relevant to the situation explored in experiments with externally driven breathing oscillations in Hall thrusters [6,9,12]. A comparison of theoretical thrust and power with experiments will be a subject of a separate paper.

## III. Modulated Thrust

For simplicity of our analysis, we assume that the oscillations in ion current and the energy are sinusoidal and of the form (Fig. 2)

$$\begin{aligned} I_i(\theta) &= I_{im} + I_{ia}\sin(\theta + \phi_i) \\ V_i(\theta) &= V_{im} + V_{ia}\sin\theta \end{aligned} \tag{6}$$

Here $I_{im}$ is mean ion current, $I_{ia}$ is amplitude of ion current oscillations, $V_{im}$ is mean ion energy, $V_{ia}$ is amplitude of ion energy oscillations, and $\phi_i$ is the phase angle between current and ion energy. Ion energy is expressed as the accelerating voltage. Thrust can be

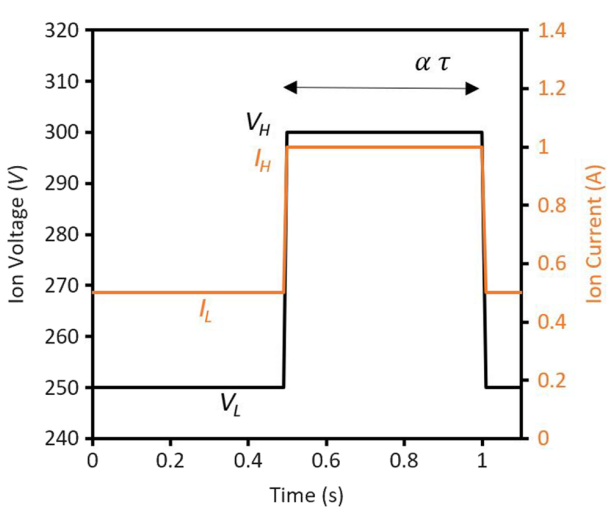


**Fig. 1 Ion voltage and ion current with time at 1 Hz pulsed operation.**

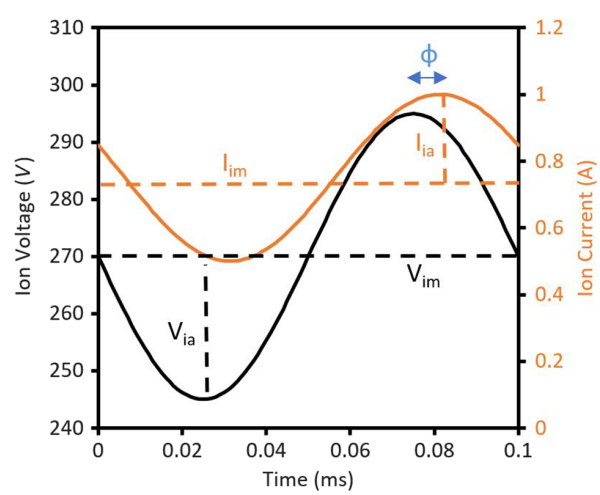


**Fig. 2 Ion voltage and ion current with time at 10 kHz sine-wave operation.**

derived by finding the time average of the product of the instantaneous ion mass flow and energy, as shown in Eq. (4). Substituting our equations for ion current and ion energy for mass flow and $v_{\mathrm{ex}}$, respectively, this expression can be solved into a form with elliptic integrals of the first and second kind. The full derivation for thrust is shown in Appendix B and comes out to

$$T_{\mathrm{mod}} = \sqrt{\frac{2MV_{im}}{e}}\left(I_{im}\left(1 - \sum_{n=1}^{\infty} a_{2n}\bar{\upsilon}^{2n}\right) + I_{ia}\cos\phi_i \sum_{n=1}^{\infty} a_{(2n-1)}\bar{\upsilon}^{(2n-1)}\right) \tag{7}$$

where $\bar{\upsilon} = V_{ia}/V_{im}$ and the coefficients $a_n$ can be found in Appendix D. A low-error first-order approximation of this expression for practical usage including associated error can also be found in Appendix B. Note that the measured energy and current oscillations of the natural breathing mode and the modulated thruster are not exactly sinusoidal in shape [10]; however, it serves as a close approximation to the these waveforms that allows analytic treatment.

The modulated thrust in Eq. (7) can be separated into the sum of three terms: the steady-state thrust of the mean voltage and current ($T_{\mathrm{steady}}$), a portion that decreases thrust ($T_D$), and a portion that increases thrust ($T_I$):

$$T_{\mathrm{mod}} = T_{\mathrm{steady}} - T_D + T_I \tag{8}$$

$$T_{\mathrm{steady}} = \sqrt{\frac{2MV_{im}}{e}} I_{im} = \dot{m}_{\mathrm{tot}} v_{\mathrm{jet}} \tag{9}$$

$$T_D = \sqrt{\frac{2MV_{im}}{e}} I_{im} \sum_{n=1}^{\infty} a_{2n}\bar{\upsilon}^{2n} \tag{10}$$

$$T_I = \sqrt{\frac{2MV_{im}}{e}} I_{ia}\cos\phi_i \sum_{n=1}^{\infty} a_{(2n-1)}\bar{\upsilon}^{(2n-1)} \tag{11}$$

Unintuitive effects of oscillation on thrust can be seen in Eq. (8) and illustrated in Fig. 3. Here plots of thrust versus ion energy amplitude are shown using typical ion energy and ion currents found in recent modulation experiments with a cylindrical Hall thruster (CHT): $I_{im} = 0.3A$, $I_{ia} = 0.3A$, and $V_{im} = 160$ V [10]. Larger modulations in ion energy $\bar{\upsilon}$ lower the thrust through $T_D$,

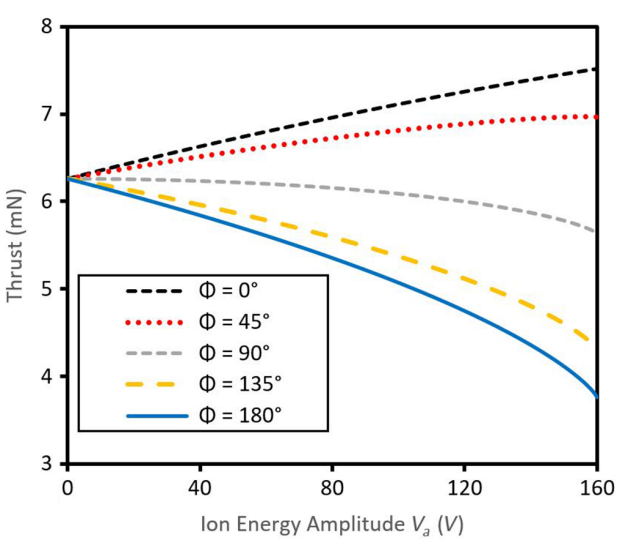


**Fig. 3 Calculated thrust vs energy amplitude over several phase angles. Thruster parameters: $V_{im} = 160$ V, $I_{im} = I_{ia} = 0.3$ A with sinusoidal modulation.**

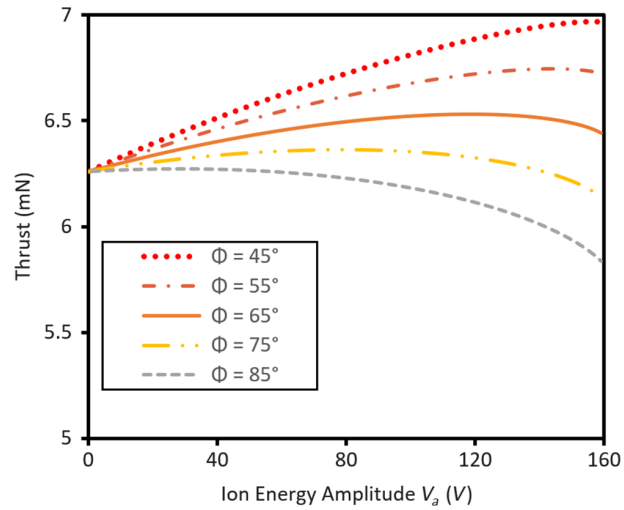


**Fig. 4 Calculated thrust vs energy amplitude from 45° to 85°. Thruster parameters: $V_{im} = 160$ V, $I_{im} = I_{ia} = 0.3A$ with sinusoidal modulation.**

but these same modulations increase the $T_I$ portion, scaled by $\cos\phi_i$. This results in a net increase in thrust at low phase angles, but a decrease at high phase.

A large phase angle implies that a higher portion of ions are accelerated at a lower voltage and contribute less to thrust. As the voltage oscillation increases, this effect gets worse, which can lower the thrust by as much as 40% below the DC level when phase is 180°. At low phase angles a higher portion of ions are accelerated at high voltage and so thrust increases. The transition region between these two effects holds some interest as the boost, or decrease in thrust is not linear with oscillation amplitude. At midrange phase angles (between roughly 45° and 85°) the thrust initially increases with ion energy amplitude before decreasing. This causes a maximum in thrust, which can be seen in Fig. 4. This nonlinearity in the thrust is dependent on the modulating waveform, and it suggests a theoretical maximum of the thrust for the thruster with modulated operation or oscillating thruster. Taking the derivative of the thrust equation [Eq. (7)] with respect to $\bar{\upsilon}$, it is possible to find an oscillating regime that would theoretically provide the maximum thrust for a given phase angle and ion current. Solving for the maximum thrust $V_{ia}$ involves finding the roots of $n$th-degree polynomials, depending on the order of expansion as shown in Eq. (C4) in Appendix C.

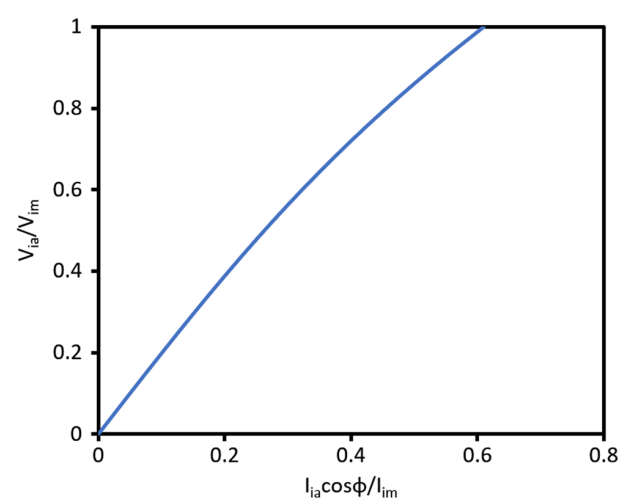


**Fig. 5 $V_{ia}/V_{im}$ that provides maximum thrust with respect to current flow parameter $I_{ia}\cos\phi_i/I_{im}$ for sinusoidal modulation.**

$$\frac{I_{ia}\cos\phi_i}{I_{im}} = \frac{\sum_{n=1}^{\infty} 2na_{2n}\upsilon^{2n-1}}{\sum_{n=1}^{\infty}(2n-1)a_{(2n-1)}\upsilon^{(2n-2)}} \tag{12}$$

Hence, the dimensionless $\bar{\upsilon}$, which provides the maximum thrust, depends solely on another dimensionless parameter: $I_{ia}\cos\phi_i/I_{im}$. Solving the relationship between the two dimensionless parameters can be done numerically. The curve on Fig. 5 corresponds to the optimal relationship between the dimensionless parameters at which a theoretical thrust maximum can be achieved for the thruster with oscillations. This theoretical maximum thrust is only applicable in a small transition regime where the term $I_{ia}\cos\phi_i/I_{im}$ is not too large. Thus it does not present the highest theoretically possible thrust for the sinusoidal waveform.

## IV. Performance

### A. Input Power

To find the efficiency of the oscillating plasma thruster, in addition to the thrust [Eq. (8)], it is also necessary to account for effects of oscillations on the input power. For example, in Hall thruster experiments with externally driven oscillations of the discharge voltage, a phase difference between the discharge current and discharge voltage was often observed [9]. Thus the expression for the input power of oscillating thruster has to be different from the expression used for conventional DC power Hall and ion thrusters.

Consider the case of a sinusoidally oscillating input voltage offset by some DC level, much like the form of ion energy and ion current.

$$\begin{aligned} I_d(\theta) &= I_{dm} + I_{da}\sin(\theta+\phi_d) \\ V_d(\theta) &= V_{dm} + V_{da}\sin\theta \end{aligned} \tag{13}$$

where $I_{dm}$ is mean discharge current, $I_{da}$ is amplitude of discharge current oscillations, $V_{dm}$ is mean discharge voltage, $V_{da}$ is the amplitude of discharge voltage oscillations, and $\phi_d$ is the phase angle between discharge current and voltage. The input power can be found by integrating the product of the discharge current and voltage, which comes out to

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2\pi}\int_0^{2\pi}(I_{dm} + I_{da}\sin(\theta+\phi_d))(V_{dm} + V_{da}\sin\theta)\,\mathrm{d}\theta \\ &= I_{dm}V_{dm} + I_{da}V_{da}\cos(\phi_d)/2 \end{aligned} \tag{14}$$

The power can then be seen as the mean component ($I_{id}V_{dm}$) of the power plus the AC component ($I_{da}V_{da}\cos(\phi_d)/2$), which is dependent on the phase angle $\phi_d$. The input electric power will be at minimum when current and voltage are out of phase and is at maximum when the two are in phase. For DC thruster operation, switch-mode power supplies are designed with large enough capacitors to reduce the voltage ripple on the thruster input to ensure $V_{da} = 0$. However, if the thruster is able to be operated with high discharge phase $\phi_d$, it may be possible to allow high-voltage ripples with minimal increases to input power and, in doing so, allow favorable alterations to the power supply design.

### B. Efficiency

Using Eq. (2), the thruster efficiency is often split into three separate components: current efficiency, voltage efficiency, and propellant utilization [13]. Other terms such as plume divergence will not be considered here. Taking the typical current efficiency ($\eta_{\text{curr}} = I_{im}/I_{dm}$), voltage efficiency ($\eta_{\text{volt}} = V_{im}/V_{dm}$), and propellant utilization ($\eta_{\text{prop}} = \dot{m}_i/(\dot{m}_i + \dot{m}_n)$), efficiency is

$$\eta = \eta_{\text{curr}}\eta_{\text{volt}}\eta_{\text{prop}} \tag{15}$$

This is a useful form of efficiency and desirable to keep. The difficulty in doing so arises from the fact that not only are ion energy and ion current oscillating in time, the discharge voltage and current may be too. The product of the mean of each of these is not equal to the mean of the product of these terms. The time-dependent effects, such as phasing differences, may alter the efficiency. However, it is possible to derive an efficiency that maintains this form with the addition of another term that includes these oscillatory effects. Starting by deriving thrust power $P_{\text{Thrust}} = T^2/(2\dot{m}_{\text{tot}})$ and continuing the derivation form of thrust described in Appendix B, where the terms A and B contain elliptic integrals,

$$\begin{aligned} P_{\text{Thrust}} &= \frac{(\overline{T})^2}{2\overline{m}_{\text{tot}}} \\ &= \frac{\left((1/\tau)\int_0^\tau \dot{m}(t)v_{\text{ex}}(t)\,\mathrm{d}t\right)^2}{2\left((1/\tau)\int_0^\tau \dot{m}(t)\,\mathrm{d}t\right)} \\ &= \frac{(A+B)^2}{4\pi^2 I_{im}} \end{aligned} \tag{16}$$

The last equation is specific for a plasma thruster with sinusoidal oscillations of ion current and ion energy. Substituting Eqs. (16) and (14) into Eq. (2), assuming only singly charged ions ($\eta_{\text{prop}} = 1$), and expanding thrust to the second order,

$$\begin{aligned} \eta &= \frac{P_{\text{Thrust}}}{P_{\text{in}}} \\ &= \frac{(A+B)^2/4\pi^2 I_{im}}{I_{dm}V_{dm} + I_{da}V_{da}\cos(\phi_d)/2} \\ &= \frac{I_{im}}{I_{dm}}\frac{V_{im}}{V_{dm}}\frac{(1-(\bar{\upsilon}^2/16) + \bar{i}\,\bar{\upsilon}\cos(\phi_i)/4)^2}{(1+\bar{i}_d\bar{\upsilon}_d\cos(\phi_d)/2)} \\ &= \eta_{\text{curr}}\eta_{\text{volt}}\frac{(1-(\bar{\upsilon}^2/16) + \bar{i}\,\bar{\upsilon}\cos(\phi_i)/4)^2}{(1+\bar{i}_d\bar{\upsilon}_d\cos(\phi_d)/2)} \\ &= \eta_{\text{curr}}\eta_{\text{volt}}\eta_{\text{osc}} \end{aligned} \tag{17}$$

We achieve a form similar to the typical efficiency equation where $\bar{i} = I_{ia}/I_{im}$, $\bar{i}_d = I_{da}/I_{dm}$, and $\bar{\upsilon}_d = V_{da}/V_{dm}$. This oscillation term $\eta_{\text{osc}}$ accounts for the phase variations both in the discharge power and ion thrust power.

$$\eta_{\text{osc}} = \frac{(1-(\bar{\upsilon}^2/16) + \bar{i}\,\bar{\upsilon}\cos(\phi_i)/4)^2}{(1+\bar{i}_d\bar{\upsilon}_d\cos(\phi_d)/2)} \tag{18}$$

From the above equation it can be seen that for an oscillating thruster the performance is dependent on both the ion phase $\phi_i$ and the discharge phase $\phi_d$. The form of Eq. (18) is specific to a sinusoidally modulating thruster with thrust expanded to the second order, but a similar equation may be found for further orders of expansion or for different waveforms. The numerator of the oscillatory $\eta_{\text{osc}}$ term alters the performance due to the alignment of the ion velocity with the ion flow. What is particularly interesting is the decrease in performance due to higher amplitude of ion energy (and ion velocity) by the second term of the numerator, even when the ion energy is in phase with the ion flow. This term occurs due to the difference between thrust power and the kinetic power of the ions in the thruster exhaust, the dispersion efficiency, and requires some analysis.

There are several efficiency terms that are not included in this analysis, namely, the plume divergence, the charge utilization (relating to multicharged ions), and the propellant utilization. These factors were primarily not included because analytic relations between the plasma oscillations and these properties are currently unknown. There is experimental evidence pointing to plume divergence increasing as discharge oscillations increase in Hall thrusters [14], a large fraction of multicharged ions in certain thrusters such as cylindrical Hall thrusters [15,16], and the knowledge that the primary mechanism of large plasma oscillations such as the breathing mode is neutral ionization, and so a full model would include these terms. As a first-order approximation, assuming that neutral

velocity does not change in time, the propellant utilization is unaffected by these plasma oscillations and could be derived as normal and included in the efficiency equation (19) [13]. This may be challenged by considering that during high plasma oscillations, only the slower neutrals are ionized. Similarly the ratio of multi-charged ions and plume angle may change during oscillations, but without any theory of how these change with plasma oscillations, no analytic relation can be derived and such considerations are beyond the scope of this work, although future studies should address this point.

$$\eta = \eta_{\text{curr}}\eta_{\text{volt}}\eta_{\text{osc}}\eta_{\text{prop}} \tag{19}$$

## V. Dispersion Efficiency

It can be shown that when the ion velocity distribution function (VDF) is described by a delta function (no energy spread), the thrust power is equal to the kinetic power of the accelerated ion flow. However, when the ion VDF is broader than the delta function, the thrust power is no longer equal to the kinetic power. Given some random VDF of the ions, the thrust power is proportional to the squared mean of the particles' velocity (jet velocity), whereas the kinetic power is proportional to the squared quadratic mean (root mean square) of the velocity. This is evident when considering the form of thrust power and kinetic power of the exhaust. Consider some velocity distribution $v$ with associated mass flow $\dot{m}$:

$$P_{\text{Kinetic}} = \frac{\int \dot{m}v^2}{2} \tag{20}$$

$$P_{\text{Thrust}} = \frac{T^2}{2\dot{m}_{\text{tot}}} = \frac{\left(\int \dot{m}v\right)^2}{2\int \dot{m}} \tag{21}$$

In any velocity distribution, the root mean square (RMS) value is always greater than or equal to the mean. A simple illustration of the divergence of these terms is shown for a Gaussian velocity distribution in Fig. 6. Proof of this can be found by the Schwartz inequality, which for the thrust and kinetic power case can be seen by multiplying each side by the total mass flow.

$$\left(\int \dot{m}v\right)^2 \leq \int \dot{m}v^2 \int \dot{m} \tag{22}$$

meaning

$$P_{\text{Thrust}} \leq P_{\text{Kinetic}} \tag{23}$$

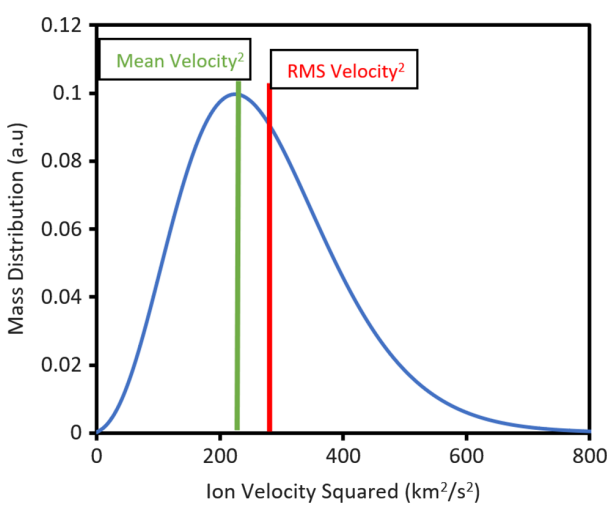


**Fig. 6 Gaussian velocity distribution with both mean velocity squared and RMS velocity squared. *X* axis is velocity squared for ease of visualization.**

Equality between these two terms can be set by assigning an efficiency factor:

$$\eta_d = \frac{P_{\text{Thrust}}}{P_{\text{Kinetic}}} \tag{24}$$

This factor was first introduced without a specific name [17], but has since had a variety of names, including the dispersion efficiency [18,19], the phi factor for VDF [20], and the propellant efficiency [21]. In this paper we shall be using the term "dispersion efficiency." The physical meaning behind the dispersion efficiency is the imbalance in kinetic energy fast and slow moving particles take compared with the thrust they provide, which results in an inefficiency of the transformed kinetic power. For a given mass flow rate and propellant utilization (i.e., ion flow), a lower input power is required to achieve the targeted thrust with mono-energetic ions with the velocity $v_{\text{jet}}$ than with ions having a VDF including ions with $v_i > v_{\text{jet}}$ and $v_i < v_{\text{jet}}$. It implies that to achieve the same thrust, the presence of slow ions ($v_i < v_{\text{jet}}$) will need to be compensated by faster ions. The generation of these faster ions will require more power than would be needed for mono-energetic ions to produce the same thrust. As a result, a wider velocity distribution will have a lower dispersion efficiency and lower efficiency. Velocity distributions are common in plasma thrusters: accelerated ions often have a low-velocity tail due to ions born downstream of the ionization region. Efficiency is then decreased not only by the lowered mean exhaust velocity (and so voltage efficiency), but also by an inefficient transformation of kinetic power to thrust power by the dispersion efficiency. It is important to distinguish the fraction of the input electric that goes to the kinetic power of the ions and the fraction of the input power that goes directly to the thrust generation. The former is defined by the current utilization efficiency times the voltage efficiency. The latter is by the inclusion of this term, which considers the velocity distribution, where the presence of slow and fast ions decreases the portion of kinetic power that is converted to thrust power.

This is concerning for oscillating thrusters as it is inherent in the device that there is a distribution in velocity because ion velocity is changing with time. Because of the smaller time scale of velocity oscillations compared with the thrust operation, the varying velocities (with time) are grouped into a single distribution of which the dispersion efficiency applies. This represents an inherent decrease in efficiency of such devices. The degree to which this is a problem for sinusoidal oscillations is analyzed in Sec. V.B. For demonstration purposes, the case of no oscillations and two species (ions and neutrals) with single velocities will be considered first, where it can be shown that the propellant utilization falls out of the dispersion efficiency. It should be noted that the dispersion efficiency is similar to the squared inverse of the "form factor" used in electrical engineering. The form factor is the ratio of RMS signal to mean signal, where in this case the form factor is of the velocity distribution.

### A. No Oscillations: Two Species

When deriving the total efficiency the dispersion efficiency is either ignored by assuming a single species of propellant, which essentially assigns the velocity distribution as a delta function and collapses the integrals to provide equality between thrust and kinetic power, or it is assumed that there are a discrete number of species. Usually this is taken to be a singly charged ion and a neutral species, which turns the dispersion efficiency into the propellant utilization. This can be seen by taking the above dispersion efficiency and assuming the velocity distribution is the sum of a delta function for each species. Taking Eq. (22) with $\dot{m} = \dot{m}_n\delta(v_n) + \dot{m}_i\delta(v_i)$, where $v_n$ and $v_i$ are the neutral and ion velocity, respectively, and $\dot{m}_n$ and $\dot{m}_i$ are the neutral and ion mass flow, respectively:

$$\begin{aligned}\left(\int \dot{m}v\right)^2 &= \left(\int (\dot{m}_n\delta(v_n) + \dot{m}_i\delta(v_i))v\,\mathrm{d}v\right)^2 \\ &= (\dot{m}_n v_n + \dot{m}_i v_i)^2 \\ &= (\dot{m}_n v_n)^2 + (\dot{m}_i v_i)^2 + 2\dot{m}_i v_i \dot{m}_n v_n\end{aligned} \tag{25}$$

$$\int \dot{m}v^2 \int \dot{m} = \int (\dot{m}_n\delta(v_n) + \dot{m}_i\delta(v_i))v^2\,\mathrm{d}v \times \int (\dot{m}_n\delta(v_n) + \dot{m}_i\delta(v_i))\,\mathrm{d}v = (\dot{m}_n + \dot{m}_i)(\dot{m}_n v_n^2 + \dot{m}_i v_i^2) \tag{26}$$

where the inequality is solved by the propellant utilization form of the dispersion efficiency by assuming $v_i \gg v_n$ and plugging Eqs. (25) and (26) into Eq. (24).

$$\eta_{\text{prop}} = \frac{(\dot{m}_n v_n)^2 + (\dot{m}_i v_i)^2 + 2\dot{m}_i v_i \dot{m}_n v_n}{(\dot{m}_n + \dot{m}_i)(\dot{m}_n v_n^2 + \dot{m}_i v_i^2)} \approx \frac{(\dot{m}_i v_i)^2}{(\dot{m}_n + \dot{m}_i)(\dot{m}_i v_i^2)} \approx \frac{\dot{m}_i \dot{m}_i v_i^2}{(\dot{m}_n + \dot{m}_i)\dot{m}_i v_i^2} \approx \frac{\dot{m}_i}{\dot{m}_i + \dot{m}_n} \tag{27}$$

Including a time dependence on the mass flow (or ion current) and velocity (or ion energy) precludes one from using the propellant utilization form of the dispersion efficiency or collapsing the integrals by delta functions. Instead the full integral must be solved.

### B. Oscillations: Single Species

As a simplification, only a single species of ions with a single velocity at any point in time is considered here. Kinetic power of the oscillating ion flow is the product of the ion current and the ion energy:

$$P_{\text{Kinetic}} = \frac{1}{2\pi}\int_0^{2\pi} (I_{im} + I_{ia}\sin(\theta + \phi_i))(V_{im} + V_{ia}\sin\theta)\,\mathrm{d}\theta = I_{im}V_{im} + I_{ia}V_{ia}\cos(\phi_i)/2 \tag{28}$$

The thrust power from a thruster with oscillations remains the same form as the no-oscillation version, where the square of the time-averaged thrust is divided by the time-averaged mass flow. This is because the spacecraft will experience the time-averaged thrust as it travels. It is in these time averages that the nuance of the dispersion efficiency is found. One intuitive result revealing the dispersion efficiency can be seen when the phase angle $\phi_i = 90°$. At high ion energy oscillations $V_{ia}$ the thrust decreases (see Fig. 3), whereas the kinetic power stays constant [see Eq. (28)]. Thus, the input electric power is being transferred into kinetic power of the ions that is not resulting in thrust power. The expression for $\eta_d$ can be very involved, depending on the order of expansion of the thrust. For this paper, we shall only expand up to the second order. Taking thrust power from Eq. (16), writing $\bar{i} = I_{ia}/I_{im}$ and through some simplification,

$$\eta_d = \frac{P_{\text{Thrust}}}{P_{\text{Kinetic}}} = \frac{(A + B)^2/4\pi^2 I_{im}}{I_{im}V_{im} + I_{ia}V_{ia}\cos(\phi_i)/2} = \frac{(1 - (\bar{v}^2/16) + \bar{i}\,\bar{v}\cos(\phi_i)/4)^2}{1 + \bar{i}\,\bar{v}\cos(\phi_i)/2} \tag{29}$$

Similar solutions can be found for other orders of expansion. Figure 7 shows that dispersion efficiency decreases as low as 73%, which represents nearly 30% of kinetic power not contributing to thrust. This is the worst-case scenario that exists for a thruster when the ion energy amplitude is equal to the ion mean energy. However, by controlling the phase of the ion energy and ion current, it is possible to increase the dispersion efficiency to 95%. This highlights the importance of ensuring that the phasing of an oscillating thruster is in the optimal regime. Following Fig. 7, it may appear that suppressing oscillations increases efficiency. However, there are many other factors in thruster operation that remain unaccounted; some researchers who tried to suppress discharge oscillations did not observe a resulting increase in efficiency [22]. This dispersion efficiency may be offset by increases in the current efficiency or through favorable phasing of the discharge current and discharge voltage, as seen in the oscillation factor. Without a full analytic model it is uncertain whether such high plasma oscillations can change efficiency, despite the clear potential to increase thrust.

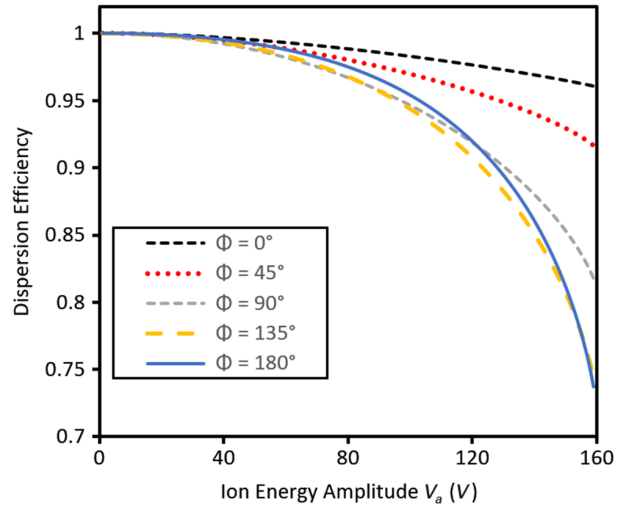


**Fig. 7 Numerically calculated dispersion efficiency vs ion energy amplitude over a range of phase angles for sinusoidal modulation. Thruster parameters: $V_{im} = 160$ V, $I_{im} = I_{ia} = 0.3$ A.**

## VI. Conclusions

The main results and following conclusions have implications for plasma thrusters operating with natural discharge oscillations and for thrusters operating with externally driven oscillations. It is shown that the thrust, and correspondingly specific impulse, can be increased with oscillations of the ion energy and the ion current. The maximum thrust is achieved when the two are in phase and oscillating with large amplitudes. A method to determine maximum thrust was shown for the out-of-phase case. It was shown that for a thruster with oscillating input voltage and current, performance is highest when the discharge current and the discharge voltage are out of phase and when ion current and ion energy are in phase. For sinusoidal oscillations, the thrust was found to potentially increase by 20% or decrease up to 40%.

Because the plasma oscillations can induce time variations of the ion VDF, the effect of the velocity distribution on the dispersion efficiency was also analyzed: a generalized form of propellant utilization. The dispersion efficiency represents the portion of kinetic power that is transformed into the thrust power, which can be decreased by a wide velocity distribution of the exhaust. This revealed an inefficiency that can significantly decrease the performance of a thruster: for sinusoidal oscillations the dispersion efficiency can be as low as 73%. By adjusting the phase between ion current and energy, however, this inefficiency can be nearly completely nullified. Although the presented analysis was conducted for a specific waveform, the same approach can be taken for any arbitrary waveform. Future work may reveal waveforms with greater gains in thrust and thruster efficiency.

## Appendix A: Square-Wave Thrust Equations

$$T=\sqrt{\frac{2M}{e}}\times\begin{cases} I_L\left(\sqrt{V_L}(1-\alpha)+\sqrt{V_H}(\alpha-\beta)\right)+I_H\sqrt{V_H}\beta, & \text{if } \alpha\geq\phi_i+\beta \\ I_L\left(\sqrt{V_L}(1-\beta-\phi_i)+\sqrt{V_H}\phi_i\right)+I_H\left(\sqrt{V_L}(\phi_i+\beta-\alpha)+\sqrt{V_H}(\alpha-\phi_i)\right), & \text{if } \phi_i+\beta\geq\alpha\geq\phi_i \\ I_L\left(\sqrt{V_L}(\phi_i-\alpha)+\sqrt{V_H}(1+\alpha-\beta-\phi_i)\right)+I_H\left(\sqrt{V_L}(1-\phi_i)+\sqrt{V_H}(\phi_i+\beta-1)\right), & \text{if } \alpha+1-\beta\geq\phi_i\geq\alpha \\ I_L\left(\sqrt{V_L}(\beta-1)\right)+I_H\left(\sqrt{V_L}(\beta-\alpha)+\sqrt{V_H}(\alpha)\right), & \text{if } \phi_i+\beta-1\geq\alpha \\ I_L\left(\sqrt{V_H}(\beta-1)\right)+I_H\left(\sqrt{V_L}(1-\alpha)+\sqrt{V_H}(\alpha+\beta-1)\right), & \text{if } \beta\geq\alpha\geq\phi_i \\ I_L\left(\sqrt{V_L}(1-\alpha-\beta)\sqrt{V_H}(\alpha)\right)+I_H\left(\sqrt{V_L}(\beta)+\sqrt{V_H}(\alpha)\right), & \text{if } 1-\beta\geq\phi_i\geq\alpha \end{cases} \tag{A1}$$

For a square-wave oscillation as shown in Fig. A1 these solutions are shown in Eq. (A1), where $\alpha$ is the duty cycle of the ion modulation, $\beta$ is the duty cycle for the ion current modulation, and $\phi_i$ is the dimensionless phase shift between the two parameters. All parameters are shown graphically in Fig. A1, and the particular duty cycle/phase-shift combination observed in the figure is described by the second case in Eq. (A1).

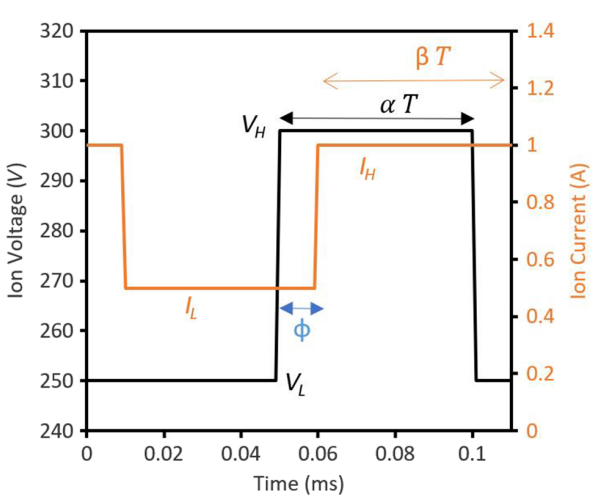


**Fig. A1 Ion voltage and ion current with time at 10 kHz pulsed operation.**

## Appendix B: Full Derivation of Thrust: Oscillations

Thrust for a thruster with oscillations is found by taking the time average of the instantaneous thrust over the oscillations, where the instantaneous thrust has the form $T=\dot{m}v_{\text{ex}}$. Here we are assuming a single species of exhaust.

$$\begin{aligned} T_{\text{mod}} &= \frac{1}{\tau}\int_0^\tau \dot{m}(t)v_{\text{ex}}(t)\,\mathrm{d}t \\ &= \frac{1}{\tau}\int_0^\tau \dot{m}(t)\sqrt{2K(t)/M}\,\mathrm{d}t \end{aligned} \tag{B1}$$

For purposes of equating the thrust and kinetic power later for electric propulsion devices, the mass flow and exhaust energy will be written as current ($I_i=\dot{m}e/M$) and voltage ($V_i=K/e$), respectively. Note that this analysis is not restricted to ion propellant, as these are only separated from mass flow and energy by a constant.

$$\begin{aligned} T_{\text{mod}} &= \frac{1}{\tau}\int_0^\tau \frac{I_i(t)M}{e}\sqrt{\frac{2eV_i(t)}{M}}\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{I_i(\theta)M}{e}\sqrt{\frac{2eV_i(\theta)}{M}}\mathrm{d}\theta \\ &= \sqrt{\frac{2M}{e}}\frac{1}{2\pi}\int_0^{2\pi} I_i(\theta)\sqrt{V_i(\theta)}\,\mathrm{d}\theta \end{aligned} \tag{B2}$$

The form of the waveform of both the mass flow and the exhaust energy is important. Here we will assume that each is an offset sinusoid as shown in Fig. 2. That is a sinusoid which has an offset larger than the amplitude such that it is never negative. To account for a possible phase shift between the energy and mass flow, which measurements of a modulated Hall Thruster have shown to exist, a phase angle $\phi_i$ is included.

$$\begin{aligned} T_{\text{mod}} &= \sqrt{\frac{2M}{e}}\frac{1}{2\pi}\int_0^{2\pi}(I_{im}+I_{ia}\sin(\theta+\phi_i))\sqrt{(V_{im}+V_{ia}\sin\theta)}\,\mathrm{d}\theta \\ &= \sqrt{\frac{2M}{e}}\frac{1}{2\pi}\left(\int_0^{2\pi} I_{im}\sqrt{(V_{im}+V_{ia}\sin\theta)}\,\mathrm{d}\theta \right. \\ &\quad + \int_0^{2\pi} I_{ia}\sin\theta\cos\phi_i\sqrt{(V_{im}+V_{ia}\sin\theta)}\,\mathrm{d}\theta \\ &\quad \left. + \int_0^{2\pi} I_{ia}\cos\theta\sin\phi_i\sqrt{(V_{im}+V_{ia}\sin\theta)}\,\mathrm{d}\theta\right) \\ &= \sqrt{\frac{2M}{e}}\frac{1}{2\pi}(A+B+C) \end{aligned} \tag{B3}$$

Each integral ($A$, $B$, and $C$) will be solved separately.

$$\begin{aligned} A &= \int_0^{2\pi} I_{im}\sqrt{(V_{im}+V_{ia}\sin\theta)}\,\mathrm{d}\theta \\ &= I_{im}\sqrt{V_{im}}\int_0^{2\pi}\sqrt{1+\frac{V_{ia}}{V_{im}}\sin\theta}\,\mathrm{d}\theta \end{aligned} \tag{B4}$$

Equation (B4) can be reduced to a form involving elliptic integrals of the second kind, $E(k)$. Taking $\bar{\upsilon}=V_{ia}/V_{im}$ as our independent variable:

$$A(\bar{\upsilon}) = 2I_{im}\sqrt{V_{im}}\left(\sqrt{1-\bar{\upsilon}}E\left(\frac{2\bar{\upsilon}}{\bar{\upsilon}-1}\right)+\sqrt{1+\bar{\upsilon}}E\left(\frac{2\bar{\upsilon}}{1+\bar{\upsilon}}\right)\right) \tag{B5}$$

The complete elliptic integral of the second kind can be expressed by the power series

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \frac{k^{2n}}{1-2n} \tag{B6}$$

Integral $A$ can then be simplified to

$$A(\bar{\upsilon}) = 2\pi I_{im}\sqrt{V_{im}}\left(1 - \frac{\bar{\upsilon}^2}{16} - \frac{15\bar{\upsilon}^4}{1024} - \frac{105\bar{\upsilon}^6}{16384} - \dots\right) \tag{B7}$$

A similar approach is taken to find integral $B$:

$$\begin{aligned} B &= \int_0^{2\pi} I_{ia}\sin\theta\cos\phi_i\sqrt{V_{im} + V_{ia}\sin\theta}\,\mathrm{d}\theta \\ &= I_{ia}\cos\phi_i\sqrt{V_{im}}\int_0^{2\pi}\sin\theta\sqrt{1 + \frac{V_{ia}}{V_{im}}\sin\theta}\,\mathrm{d}\theta \end{aligned} \tag{B8}$$

Again taking $\bar{\upsilon} = V_{ia}/V_{im}$ as our independent variable, the solution of Eq. (B8) can be solved into a form involving the elliptic integrals of both the first $K(k)$ and second kind $E(k)$.

$$\begin{aligned} B(\bar{\upsilon}) = {} & \frac{2I_{ia}\cos\phi_i\sqrt{V_{im}}}{3\bar{\upsilon}} \\ & \times\left(\sqrt{1-\bar{\upsilon}}\left(E\left(\frac{2\bar{\upsilon}}{\bar{\upsilon}-1}\right) - (\bar{\upsilon}+1)K\left(\frac{2\bar{\upsilon}}{\bar{\upsilon}-1}\right)\right)\right. \\ & \left. + \sqrt{\bar{\upsilon}+1}\left(E\left(\frac{2\bar{\upsilon}}{\bar{\upsilon}+1}\right) - (\bar{\upsilon}-1)K\left(\frac{2\bar{\upsilon}}{\bar{\upsilon}+1}\right)\right)\right) \end{aligned} \tag{B9}$$

Solving for the power series of Eq. (B9) provides a simplified form that quickly converges.

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n} \tag{B10}$$

$$B(\bar{\upsilon}) = 2\pi I_{ia}\cos\phi_i\sqrt{V_{im}}\left(\frac{\bar{\upsilon}}{4} + \frac{3\bar{\upsilon}^3}{126} + \frac{35\bar{\upsilon}^5}{4096} + \dots\right) \tag{B11}$$

Integral $C$ is simply zero, which can be shown by method of $u$ substitution. Taking $u = V_{im} + V_{ia}\sin\theta$:

$$\begin{aligned} C &= \int_0^{2\pi} I_{ia}\cos\theta\sin\phi_i\sqrt{V_{im} + V_{ia}\sin\theta}\,\mathrm{d}\theta \\ &= I_{ia}\sin\phi_i\int_0^{2\pi}\cos\theta\sqrt{V_{im} + V_{ia}\sin\theta}\,\mathrm{d}\theta \\ &= I_{ia}\sin\phi_i\int_{\theta=0}^{\theta=2\pi}\frac{\cos\theta\sqrt{u}}{V_{ia}\cos\theta}\,\mathrm{d}u \\ &= \frac{I_{ia}\sin\phi_i}{V_{ia}}\int_{\theta=0}^{\theta=2\pi}\sqrt{u}\,\mathrm{d}u \\ &= \frac{2I_{ia}\sin\phi_i}{3V_{ia}}((V_{im} + V_{ia}\sin 2\pi)^{3/2} - (V_{im} + V_{ia}\sin 0)^{3/2}) \\ &= 0 \end{aligned} \tag{B12}$$

For a thruster with an oscillation in the mass flow and energy of an offset-sinusoid form, the thrust can then be written as

$$\begin{aligned} T_{\mathrm{mod}} = {} & \sqrt{\frac{2MV_{im}}{e}}\left(I_{im}\left(1 - \sum_{n=1}^{\infty} a_{2n}\bar{\upsilon}^{2n}\right)\right. \\ & \left. + I_{ia}\cos\phi_i\sum_{n=1}^{\infty} a_{(2n-1)}\bar{\upsilon}^{(2n-1)}\right) \end{aligned} \tag{B13}$$

where the coefficients $a_n$ can be found through the power series of the elliptic integrals. The first six terms are shown in Eqs. (B7) and (B11). When there is no oscillation in ion energy or when the expansion is taken to the zeroth order, Eq. (B13) reduces to $T = \sqrt{(2MV_{im}/e)}I_{im} = v_{\mathrm{jet}}\dot{m}_{\mathrm{tot}}$.

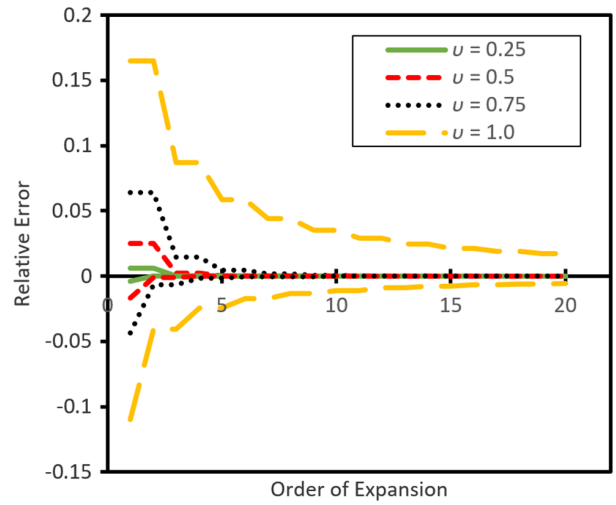


**Fig. B1 Error of integral $A$ (positive) and $B$ (negative) vs the expansion order over a range of $V_{ia}/V_{im}$.**

Although it is possible to solve the modulated thrust for expansion orders to the nth degree, the series quickly converges, particularly for lower $\bar{\upsilon} = V_{ia}/V_{im}$. The error of the series expansion for integral $A$ and integral $B$ is shown in Fig. B1. Error is defined here as the difference between the series expansion and the numerically calculated divided by the numerically calculated. Figure B1 shows the error below 2% at third order for $\bar{\upsilon} < 0.5$. For thrusters with much fuller oscillations with $\bar{\upsilon}\sim 1$, error on thrust is less than 5% with sixth-order expansion. Several useful forms of the thrust follow:

If the ion energy oscillation amplitude is less than half the mean ion energy the thrust can be expanded to the first order with error below 4%:

$$T_{\mathrm{mod}} \approx \sqrt{\frac{2MV_{im}}{e}}\left(I_{im} + I_{ia}\cos\phi_i\frac{V_{ia}}{4V_{im}}\right) \tag{B14}$$

If the ion energy oscillation amplitude is equal to the mean ion energy (full pulse), an exact expression can be found:

$$T_{\mathrm{mod}} = \frac{4}{3\pi}\sqrt{\frac{MV_{im}}{e}}(3I_{im} + I_{ia}\cos\phi_i) \tag{B15}$$

## Appendix C: Maximum Thrust Derivation

To derive voltage amplitude that will provide the maximum thrust, we take our derived expression of thrust [Eq. (7)] and take the derivative with respect to $\bar{\upsilon}$.

$$\begin{aligned} \frac{dT_{\mathrm{mod}}}{d\bar{\upsilon}} = {} & \sqrt{\frac{2MV_{im}}{e}}\left(I_{im}\left(-\sum_{n=1}^{\infty} 2na_{2n}\bar{\upsilon}^{2n-1}\right)\right. \\ & \left. + I_{ia}\cos\phi_i\sum_{n=1}^{\infty}(2n-1)a_{(2n-1)}\bar{\upsilon}^{(2n-2)}\right) \end{aligned} \tag{C1}$$

We then set the left-hand side of Eq. (C1) to zero and simplify.

$$\begin{aligned} 0 = {} & \sqrt{\frac{2MV_{im}}{e}}\left(I_{im}\left(-\sum_{n=1}^{\infty} 2na_{2n}\bar{\upsilon}^{2n-1}\right)\right. \\ & \left. + I_{ia}\cos\phi_i\sum_{n=1}^{\infty}(2n-1)a_{(2n-1)}\bar{\upsilon}^{(2n-2)}\right) \end{aligned} \tag{C2}$$

$$I_{im}\sum_{n=1}^{\infty} 2na_{2n}\bar{\upsilon}^{2n-1} = I_{ia}\cos\phi_i\sum_{n=1}^{\infty}(2n-1)a_{(2n-1)}\bar{\upsilon}^{(2n-2)} \tag{C3}$$

$$\frac{\sum_{n=1}^{\infty} 2na_{2n}\bar{v}^{2n-1}}{\sum_{n=1}^{\infty}(2n-1)a_{(2n-1)}\bar{v}^{(2n-2)}} = \frac{I_{ia}\cos\phi_i}{I_{im}} \tag{C4}$$

This expression can then be numerically solved for $\bar{v}$, as was performed in Fig. 5.

## Appendix D: Thrust Expansion Coefficients

The coefficients $a_n$ for the series expansion in Eq. (B13) are shown up to 12th order. These coefficients can be calculated through the series expansion of integrals $A$ and $B$.

$$\begin{aligned}
a_1 &= \frac{1}{4} = 0.2500 \\
a_2 &= \frac{1}{16} = 0.0625 \\
a_3 &= \frac{3}{128} = 0.0234 \\
a_4 &= \frac{15}{1024} = 0.0146 \\
a_5 &= \frac{35}{4096} = 0.0085 \\
a_6 &= \frac{105}{16384} = 0.0064 \\
a_7 &= \frac{1155}{262144} = 0.0044 \\
a_8 &= \frac{15015}{4194304} = 0.0036 \\
a_9 &= \frac{45045}{16777216} = 0.0027 \\
a_{10} &= \frac{153153}{67108864} = 0.0023 \\
a_{11} &= \frac{969969}{536870912} = 0.0018 \\
a_{12} &= \frac{6789783}{4294967296} = 0.0016
\end{aligned} \tag{D1}$$

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